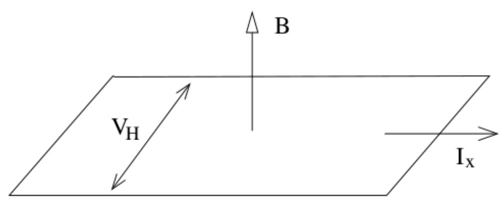
The Chern-Simons Theory and Quantum Hall Effect

Abstract

This page contains a brief review of my understanding of the quantum Hall effect. These notes are written in Notion, it will be converted to LaTeX soon.

Classical Hall Effect

The classical Hall Effect consists of a plane with a current I in the $\hat{x}$ direction. A magnetic field $\vec{B}$ is applied in the $\hat{z}$ direction consequently bending the current towards the $\hat{y}$ direction and producing a voltage V_H, where H states for Hall, just like the figure.



The electrons in this system move in a circular motion due to the magnetic field $\vec{B}$, with a frequency given by

$$\omega_c = \frac{eB}{m}.$$

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Drude Model

Suppose we add an electric field $\vec{E}$ whose will be responsible for accelerate the electrons in the absence of the magnetic field. We might consider the collisions that the electrons may have along the material with whatever can impede them. This is called the *Drude Model* and it is a first attempt to describe a conductor as if it was billiard balls. The equation of motion of this system is given by

$$m\frac{d\vec{v}}{dt} = -e\vec{E} - e\vec{v} \times \vec{B} - \frac{m\vec{v}}{\tau}.$$

We want the stationary solutions, that is,

$$\frac{d\vec{v}}{dt} = 0.$$

So we must solve the equation

$$e\vec{E} + e\vec{v} \times \vec{B} + \frac{m\vec{v}}{\tau} = 0.$$

Recalling that the current density is related to the velocity by

$$\vec{J} = -ne\vec{v}.$$

In matrix notation

$$\begin{pmatrix} 1 & \omega_c \tau \\ -\omega_c \tau & 1 \end{pmatrix} \vec{J} = \frac{e^2 n \tau}{m} \vec{E}.$$

And if we invert this relation

$$\vec{J} = \sigma \vec{E}.$$

This *Ohm's Law* that states how the current behaves in the presence of a electric field. We've defined

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix},$$

as the *conductivity tensor.* This is something new that appears only in the presence of a magnetic field. Just as for the scalar case, the resistivity is the inverse of the conductivity, therefore

$$\rho = \sigma^{-1}.$$

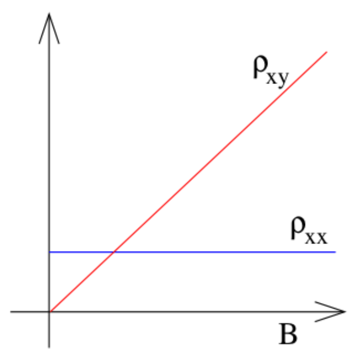
More explicitly

$$\rho = \frac{1}{\sigma_{DC}} \begin{pmatrix} 1 & \omega_c \tau \\ -\omega_c \tau & 1 \end{pmatrix}.$$

The off-diagonal terms are responsible for the Hall effect. It can be shown that

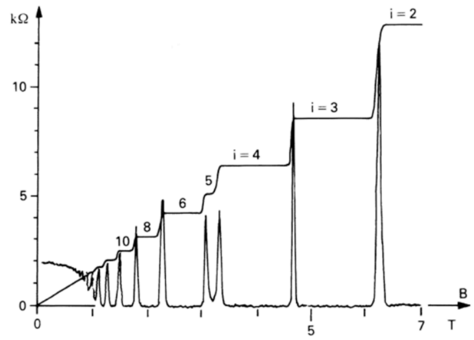
$$\rho_{xx} = \frac{m}{ne^2 \tau}, \quad \rho_{xy} = \frac{B}{ne},$$

that graphically is represented as



Integer Quantum Hall Effect

The quantum Hall effect was first discovered experimentally. It can be achieved by turning on a strong magnetic field and goin to low temperatures. The graph for ρ_{xx} and ρ_{xy} are shown below



The plateux represent the transverse resistivity ρ_{xy} and the spikes are the longitudinal resistivity ρ_{xx}. Note that it is interesting because the resistivity over the plateux has the value

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu},\ \nu \in \mathbb{Z}.$$

The center of these plateux occurs when the magnetic field assumes the values

$$B = \frac{2\pi\hbar n}{\nu e} = \frac{n}{\nu}\Phi_0.$$

In contrast, the values of ρ_{xx} are zero most of the time, except when ρ_{xy} changes it's value, then it spikes. What is interesting about this effect is that the resistivity is a macroscopic quantity that is quantized rather a microscopic one.

Note another thing that is very interesting. In components we have

$$\sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2}, \quad \sigma_{xy} = \frac{-\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2}.$$

If $\rho_{xy} = 0$ we have

$$\sigma_{xx} = \frac{1}{\rho_{xx}}.$$

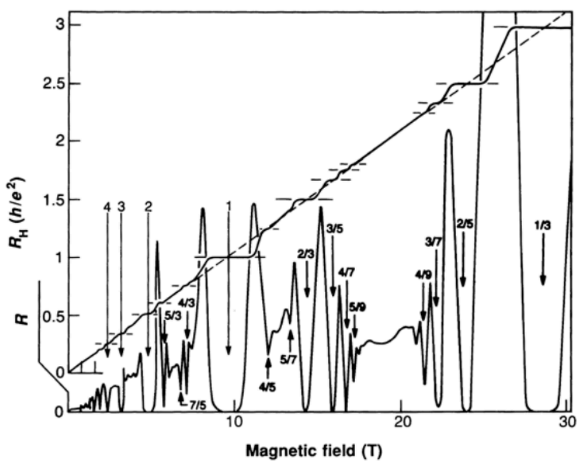
Which is a result that we already familiar with. But in the case $\rho_{xx} = 0$ with $\rho_{xy} \neq 0$ we have

$$\sigma_{xx} = 0.$$

Apparently this case shows that our system is a perfect conductor and a perfect insulator simultaneously. But these are just names! What these expressions are really telling us is that, for $\sigma_{xx} = 0$, no current if flowing in the longitudinal directions (just like an insulator) and ρ_{xx} is telling us that no energy is being dissipated (just like a conductor).

Fractional Quantum Hall Effect

The most notable difference in this effect is that now our resistivities looks like this



Additionally, we have that

$$\nu \in \mathbb{Q}.$$

One thing that is very interesting is that if start to decrease the dirty in our system, the plateux will increase. Thus, in the limit that our system is completely clean, we will get the classical results for the resistivities.

The integer quantum hall effect can be understood using free electrons, while the fractional quantum hall effect can be understood only introducing interactions.

Chern-Simons Theory: General Aspects

One way we can describe the Quantum Hall Effect (QHE) and obtain these results we've talk about is using the Chern Simons Theory.

The Chern Simons Theory is a gauge theory only possible in 3 (or in odd) dimensions. It works this way because the new term that we may add to the action is

$$S_{CS} = \frac{k}{4\pi} \int d^3x \; \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho.$$

Note that indeed this theory won't work in 3+1, for example, because of the Levi-Civita symbol. The indices doesn't match. The constant k is called the *level* of the Chern-Simons term.

At first glance, the Chern-Simons action doesn't seem gauge invariant because it depends explicitly on the field A_μ. But upon a gauge transformation

$$S_{CS} \to S_{CS} + \frac{k}{4\pi} \int d^3x \; \partial_\mu (\omega \epsilon^{\mu\nu\rho} \partial_\nu A_\rho).$$

Thus, it changes by a total derivative. For some topologies, we can simply throw this term away. But in general the border will have physical contributions.

The Chern-Simons equations of motion are given by

$$\frac{k}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho} = 0.$$

This shows that this theory has a trivial dynamics, since the solutions are just $F_{\nu\rho} = 0$.

Note that the Chern-Simons term breaks parity. In 2+1 dimensions the parity is defined as

$$x^0 \to x^0,\ x^1 \to -x^1,\ x^2 \to x^2.$$

Because if we were to define just like in 4 dimensions, that is $\vec{x} \to -\vec{x},$ since the matrix is 3-dimensional, it will have determinant 1 and therefore would be a rotation. Therefore we define that way. Note that the axis that we chose to be reflected is completely arbitrary and would work as well had we chose x^2 to be reflected. Upon those transformations, the gauge field transforms as

$$A_0 \to A_0,\ A_1 \to -A_1,\ A_2 \to A_2.$$

The integration measure $\int\ d^3x$ is invariant under parity because even if $x_1 \to -x_1$, the integration limits also change. But the integrand isn't invariant, that is

$$\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \to -\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho.$$

Thus the Chern-Simons term can only arise in theories that break parity.

We know that the conserved current couples with the conserved current just like

$$S = \int d^3x\ J^\mu A_\mu.$$

This shows then that

$$\frac{\delta S}{\delta A_\mu} = J^\mu.$$

From that, is easy to see that

$$J_i = -\frac{k}{2\pi}\epsilon_{ij}E_i.$$

Recalling that

$$\vec{J} = \sigma\vec{E},$$

we then conclude

$$\sigma_{xy} = \frac{k}{2\pi}.$$

This coincides with the Hall conductivity if we identify

$$k = \frac{e^2\nu}{\hbar}.$$

But there's nothing that tell us that ν should be quantized. We will see how this works shortly.

There is another very interesting aspect of this theory, it happens when we couple it with the Maxwell action,

$$\mathcal{L}_{\text{MCS}} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{k}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho.$$

The equations of motion for this system are

$$\partial_\mu F^{\mu\nu} + \frac{ke^2}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0.$$

Introducing the dual field

$$\tilde{F^\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} \Rightarrow \tilde{F_\mu} = \frac{1}{2}\epsilon_{\mu\alpha\beta}F^{\alpha\beta},$$

we can invert the relation by multiplying by a Levi-Civita both sides, which gives

$$F^{\sigma\rho} = \epsilon^{\mu\sigma\rho}\tilde{F_\mu}.$$

Then we get

$$\epsilon^{\alpha\mu\nu}\partial_\mu\tilde{F_\alpha} + ke^2\tilde{F^\nu} = 0.$$

This can be re-written as

$$(\partial_\mu\partial^\mu + (ke^2)^2)\tilde{F^\sigma} = 0.$$

This is the equation of motion for a field with mass $m = ke^2$. Another way to see this is by rewriting the Lagrangian as

$$\mathcal{L}_{\text{MCS}} = A_\mu\left[\frac{1}{2e^2}g^{\mu\nu}\Box - \frac{k}{2}\epsilon^{\mu\rho\nu}\partial_\rho - \frac{1}{2e^2}\left(\frac{1}{\xi} - 1\right)\partial^\mu\partial^\nu\right]A_v,$$

with a gauging fixinh term $-\frac{1}{2\xi e^2}(\partial_\mu A^\mu)^2$. This way we identify the term between brackets as the inverse of the propagator,

$$(\Pi^{-1})^{\mu\nu} = \frac{1}{2e^2}g^{\mu\nu}\Box - \frac{k}{2}\epsilon^{\mu\rho\nu}\partial_\rho - \frac{1}{2e^2}\left(\frac{1}{\xi} - 1\right)\partial^\mu\partial^\nu.$$

In the momentum space this is written as

$$(\Pi^{-1}(p))^{\mu\nu} = \frac{1}{2e^2}g^{\mu\nu}p^2 + \frac{ik}{2}\epsilon^{\mu\nu\rho}p_\rho + \frac{1}{2e^2}\left(1 - \frac{1}{\xi}\right)p^\mu p^\nu.$$

To compute the propagator, first we introduce the following operators,

$$\omega^{\mu\nu} \equiv \frac{p^\mu p^\nu}{p^2},\ S^{\mu\nu} \equiv i\epsilon^{\mu\nu\rho}p_\rho,\ \theta^{\mu\nu} = g^{\mu\nu} - \omega^{\mu\nu},$$

whose have a closed algebra

$$\begin{gathered} \omega^2 = 0, \quad \omega\theta = 0, \\ \theta^2 = 0, \quad \omega S = 0, \\ S^2 = -p^2\theta,\ \theta S = S. \end{gathered}$$

Then we write Π^{-1} and Π in terms of this basis,

$$\Pi_{\mu\nu}(p) = A\omega_{\mu\nu} + B\theta_{\mu\nu} + CS_{\mu\nu}.$$

Solving

$$\Pi_{\mu\alpha}(\Pi^{\alpha\nu})^{-1} = \delta_\mu{}^\nu$$

for A, B and C, we get

$$\Pi_{\mu\nu} = e^2 \left(\frac{p^2 g_{\mu\nu} - p_\mu p_\nu - ike^2 \epsilon_{\mu\nu\rho} p^\rho}{p^2(p^2 - k^2 e^4)} + \xi \frac{p_\mu p_\nu}{p^4} \right).$$

The pole of this propagator is interpreted as the mass of this gauge field, which is again $(ke^2)^2$.

This shows that the coupling with the Chern-Simons term worked as some kind of "Higgs mechanism", giving a mass to the photon. We call this a "topological" mass due to the nature of the Chern-Simons theory that we'll see soon. But note that we can work as well with the Higgs mechanism. This gives the Maxwell-Chern-Simons-Higgs Lagrangian

$$\mathcal{L}_{\text{MCSH}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{k}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + (\mathcal{D}_\mu \phi)^* \mathcal{D}^\mu \phi - V(|\phi|),$$

where $V(|\phi|)$ is some symmetry-breaking potential with non-trivial minimum $\langle \phi \rangle = v$. This implies a redefinition of the field

$$\phi \to \phi - v,$$

in order to this expectation value to vanish. Thus the Lagrangian becomes

$$\mathcal{L}_{\text{MCSH}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{k}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + (\mathcal{D}_\mu \phi)^* \mathcal{D}^\mu \phi + e^2 v^2 A_\mu A^\mu.$$

Thus we see that there is an additional term due to the symmetry breaking. If we use the same algorithm that we've used to the Maxwell-Chern-Simons Lagrangian, the propagator for the gauge field will be

$$\Pi_{\mu\nu} = \frac{e^2(p^2 - m_H^2)}{(p^2 - m_+^2)(p^2 - m_-^2)} \left[g_{\mu\nu} - \frac{p_\mu p_\nu}{(p^2 - \xi m_H^2)} - i \frac{ke^2 \epsilon_{\mu\nu\rho} p^\rho}{(p^2 - m_H^2)} \right] + e^2 \xi \frac{p_\mu p_\nu (p^2 - k^2 e^4 - m_H^2)}{(p^2 - m_+^2)(p^2 - m_-^2)(p^2 - \xi m_H^2)}.$$

where $m_H^2 = 2e^2 v^2$ is the Higgs mass and

$$m_\pm^2 = m_H^2 + \frac{(ke^2)^2}{2} \pm \frac{ke^2}{2} \sqrt{k^2 e^4 + 4m_H^2},$$

are the masses of the gauge field. So what we have here is the following. In the unbroken vacumm, the complex scalar field has two massive degress of freedom and the gauge field has one massive degree of freedom arising from the Chern-Simons term. In the broken vacuum, one component of the scalar field combines with the longitudinal degree of freedom of the gauge field, resulting in one massive degree of freedom from the scalar field and two massive degrees of freedom from the gauge field.

Therefore, the photon has acquired a topological mass due to the Chern-Simons term and a Higgs mass due to the Higgs mechanism.

Note that we can also work with a Chern-Simons-Higgs Lagrangian only if we take the limit

$$e \to \infty, \quad k = \text{fixed}.$$

In this case, the Lagrangian becomes

$$\mathcal{L}_{\text{CSH}} = \frac{k}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_v A_\rho + (\mathcal{D}_\mu \phi)^* \mathcal{D}^\mu \phi - V(|\phi|).$$

To compute the degrees of freedom we can take this limit in the Maxwell-Chern-Simons-Higgs Propagator, leading to

$$\Pi_{\mu\nu} = \frac{1}{p^2 - \left(\frac{2v^2}{k}\right)^2}\left[\frac{2v^2}{k}g_{\mu\nu} - \frac{p_\mu p_\nu}{2v^2} + \frac{i}{k}\epsilon_{\mu\nu\rho}p^\rho\right],$$

in which we indetify one mass pole at $p^2 = \left(\frac{2v^2}{k}\right)^2$. Thus, in the unbroken vacuum the gauge field is massless and the scalar field has two massive degrees of freedom. In the broken vacuum, just as before, one of the components of the scalar field combines with the longitudinal component of the gauge field giving a mass $\frac{2v^2}{k}$ to it. Note that we can rewrite $m_\pm$ as

$$m_\pm = \frac{ke^2}{2}\left(\sqrt{1 + \frac{8v^2}{(ke)^2}} \pm 1\right).$$

Taking the limit $e^2 \to \infty$,

$$m_\pm \sim \frac{ke^2}{2}\left(1 + \frac{4v^2}{(ke)^2} \pm 1\right) = \frac{ke^2}{2} + \frac{2v^2}{k} \pm \frac{ke^2}{2}.$$

\ Thus we have that $m_+ \to \infty$ and $m_- \to 2v^2/2,$ highlighting that indeed the gauge field has only one massive degree of freedom.

Quantization of Chern Simons Level

We've saw above that the Chern-Simons theory describes the correct expression for the Hall conductivity but there was nothing telling us that the ν should be quantized. We now proceed to show this.

The quantization of the Chern-Simons level is not easy to perform given the fact that the Chern-Simons Theory is a constrained theory and therefore it should be quantized using Dirac brackets.

Nonetheless we can obtain the results for the QHE without having to enter a heavy mathematical formalism.

Recall that we can map the statistical mechanics partition function $Z[\beta]$ by the Wick rotation $\tau = it$. That way we introduce this time with periodicity

$$\tau = \tau + \beta,$$

and now we have the concept of temperature as well. So what we've done is to introduce a time that is a S^1, modifying our space to be

$$\mathbb{R}^2 \times S^1.$$

For simplicity, consider our space is a S^2 rather than a $\mathbb{R}^2$ (these two are equivalent in the thermodynamic limit). Another demand is that there exists a magnetic flux through the S^2 given by

$$\int_{S^2} d\vec{a} \cdot \vec{B} = g.$$

We can think this as the flux due to a magnetic monopole. From the Dirac quantization condition,

$$eg = 2\pi\hbar n,\ n \in \mathbb{Z}.$$

Thus the minimum flux allowed by this condition is with $n = 1$. Therefore, in terms of the electromagnetic tensor,

$$\int_{S^2} F_{12} = \frac{2\pi\hbar}{e}.$$

The reason we do this is that, this theory should hold even in the presence of magnetic monopole, that is something reasonable to think. But we'll see that considering this, our calculations will be a lot easier.

Because of the topology of the periodicity of our time, we can do something interesting with our gauge transformations,

$$A_\mu \to A_\mu + \partial_\mu \omega.$$

We can think of ω to be periodic

$$\omega = \frac{2\pi\hbar\tau}{e\beta}.$$

That way, the gauge transformation for A_0 is given by

$$A_0' = A_0 + \frac{2\pi\hbar}{e\beta}.$$

These are called *large gauge transformations* because it can't be reduced to the identity. Then, the Chern-Simons actions explicitly gives

$$\begin{aligned} S_{CS} &= \int d^3x \ A_0 F_{12} + A_1 F_{20} + A_2 F_{01} \\ &= \int d^3x \ A_0 F_{12} + A_1(\partial_2 A_0 - \partial_0 A_2) + A_2(\partial_0 A_1 - \partial_1 A_0). \end{aligned}$$

Since the gauge field is a background field, it has no dynamics, then we can drop all time derivatives. Considering the $A_0 = a$ a constant, and integrating by parts the spatial derivatives, we get

$$S_{CS} = \frac{ka}{2\pi} \int d^3x \ F_{12}.$$

Using the Dirac quantization condition (recall that we have to change to spherical coordinates),

$$S_{CS} = \beta a \frac{\hbar k}{e}.$$

But as we've seen, under gauge transformations, the action changes as

$$\begin{aligned} S_{CS} &\to S_{CS} + \frac{k}{4\pi} \int d^3x \ \partial_\mu(\omega \epsilon^{\mu\nu\rho} \partial_\nu A_\rho) \\ &= S_{CS} + \frac{2\pi\hbar k}{e^2}, \end{aligned}$$

which highlight the fact the actions isn't invariant under gauge transformations.

However, in QFT the partition function is a more fundamental object than the action, in contrast with quantum mechanics. Therefore it's all good with the action changing under gauge transformations as long as the partition functions doesn't change. Recalling that

$$Z[A_\mu] = e^{iS[A_\mu]/\hbar},$$

we see that it is invariant if

$$\frac{\hbar k}{e^2} \in \mathbb{Z}.$$

But we've saw that

$$k = \frac{e^2 \nu}{\hbar}.$$

Therefore, ν is indeed an integer. So we've successfully described the integer QHE using Chern-Simons theory. We now proceed to describe the fractional QHE.

Fractional Quantum Hall Effect

To describe the fractional QHE we introduce a emergent gauge field a_μ associated to the collective motion of underlying electrons. This is different from the gauge field A_μ of electromagnetism. The simplest term we can add to the action involving this field is the Chern-Simons term,

$$S_{CS} = \frac{k}{4\pi} \int d^3x \ \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho.$$

As we've saw before, the Chern-Simons term isn't dynamical, but it has a *topological* contribution. One may note this by comparing with the Maxwell action, which can be written in terms of differential forms as

$$S_M = -\frac{1}{4e^2} \int F \wedge \star F.$$

To take the dual $\star F$ we need to use a metric, whose has information about the geometry of the space. But the Chern-Simons term is

$$S_{CS} = \frac{k}{4\pi} \int a \wedge da.$$

Note that this term doesn't involve any metric. Thus all the information contained in this term is topological.

Despite the fact that the field a_μ isn't the Maxwell field, it follows the same properties that we've saw before. It has its own dynamics given by

$$S[a_\mu] = -\frac{1}{4g^2} \int d^3x \ f_{\mu\nu} f^{\mu\nu},$$

and it works as a topological mass generator as well.

Effective Theory of the Laughlin States

The states that describe the fractional QHE are the Laughlin states. We now will write down an action that is able to describe those states and therefore reproduce the expected values for the Hall conductivity. First we need a coupling between the fields A_μ and a_μ. In general the coupling between involving gauge fields is trough a conserved current. Luckily we have a perfect conserved current to couple with, it is

$$J^\mu = \frac{e^2}{2\pi\hbar} \epsilon^{\mu\nu\rho} \partial_\nu a_\rho,$$

where its conservation result from the contraction of the two derivatives with the Levi-Civita symbol. We can interpret this as if the magnetic flux of a_μ is the electric charge that couples with A_μ. This magnetic flux also follows the Dirac quantization condition

$$\frac{1}{2\pi} \int_{S^2} f_{12} = \frac{\hbar}{e}.$$

This ensures that the minimum charge allowed is $\int J^0 = e$, as it should be. The effective action is then

$$S_{eff}[A;a] = \frac{e^2}{\hbar}\int d^3x\ \frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho - \frac{m}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \ldots\ .$$

The first term is the coupling and the second is the simplest term we may add involving a_μ. The other terms that may arise vanish at large distances and therefore we don't need them to our conclusions. As we've saw for the integer QHE, m must be an integer. We can write down the equation of motion for a_μ which is

$$f_{\mu\nu} = \frac{1}{m}F_{\mu\nu},$$

whose has the solution

$$a_\mu = \frac{1}{m}A_\mu.$$

Substituting back in the action we ger

$$S_{eff}[A] = \frac{e^2}{2\pi}\int d^3x\ \frac{1}{4\pi m}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho.$$

But using the same procedure that we've used before, we get the Hall conductivity

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\frac{1}{m},$$

which is the correct answer.

There is some subtleties about this derivation because the effective action has some troubles when integrating over a S^2. Moreover, the fields are constrained by the Dirac quantization condition and the equation of motion is not satisfied when both have a single unit of flux. But these subtleties lies in the manifold structure of our space and it has to do with the choice of charts, something that we won't do.