Notes on Higher-Form Symmetries

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Abstract

These notes present my current understanding of higher-form symmetries and are intended for pedagogical purposes. They are continuously updated as I learn more about the subject and are based on the references listed below.

Contents

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1 Introduction

Higher-form symmetries generalize the conventional notion of symmetry by acting on extended operators, such as lines or surfaces, rather than local, point-like operators. We do this by assigning a topological meaning to the symmetry operators and understanding its action in terms of the link between the manifold where the symmetry operator is defined and the manifold where the charged operator is defined. The consequences are that the conserved currents will be higher degree forms, rather than the usual 1-form conserved currents. We will start by assigning a topological meaning to the "usual" symmetry operators and then generalizing these ideas to extended operators.

2 Symmetry Operators as Topological Operators

We've saw that in terms of differential forms, the conservation law is written as

$$d \star j = 0, \tag{1}$$

where j is the current 1-form and $\star : \Lambda^p \to \Lambda^{D-p}$ is the Hodge star map that take a p-form and associates it to a $(D - p)$-form. Thus the conserved charge was defined as

$$Q \equiv \int d^{D-1}x \, j^0. \tag{2}$$

In terms of differential forms, we may write as

$$Q = \int_{\Sigma_{D-1}} \star j, \tag{3}$$

where Σ_{D-1} is a $D-1$ manifold. We say then that those charges are integrated over a manifold of codimension 1, which is a submanifold of a manifold of dimension D. To understand how these operators act over the fields, we will make use of the Ward identity,

$$\partial_\mu \langle j^\mu(x)\phi(y)\rangle = -i\delta^{(D)}(x - y)\langle \delta\phi(y)\rangle. \tag{4}$$

Integrating over a D-dimensional manifold Ω with boundary Σ_{D-1}, the left-hand side becomes

$$\int_\Omega d\langle \star j\phi(y)\rangle = \int_{\Sigma_{D-1}} \langle \star j\phi(y)\rangle = \langle Q(\Sigma_{D-1})\phi(y)\rangle, \tag{5}$$

where we used Stokes theorem. For the right-hand side, we have

$$-i \int_\Omega d^D x \, \delta^{(D)}(x - y)\langle \delta\phi(y)\rangle, \tag{6}$$

where we identify,

$$\int_\Omega d^D x \, \delta^{(D)}(x - y) = \begin{cases} 0, & y \in \Omega \\ 1, & y \notin \Omega \end{cases} \tag{7}$$

as the linking between the point y and the manifold Ω. Thus we call

$$\int_\Omega d^D x \, \delta^{(D)}(x - y) \equiv \text{Link}(\Sigma_{D-1}, y). \tag{8}$$

Hence

$$\langle Q(\Sigma_{D-1})\phi(y)\rangle = -i\text{Link}(\Sigma_{D-1}, y)\langle \delta\phi(y)\rangle. \tag{9}$$

What this expression is telling us is that the charge acts on the local field and it only have non-vanishing action if the point where the local operator is defined links with the manifold in which the charge is defined.

The link is unaffected by deformations as longs as they do not cross the point y. To see why, suppose that we perform a deformation of the type $\Sigma' = \Sigma + \Sigma_0$, $y \notin \Omega_0$. Then

$$Q(\Sigma + \partial\Omega_0) = \int_{\Omega \cup \Omega_0} \langle d \star j\phi(y) \rangle = \int_{\Omega} \langle d \star j\phi(y) \rangle + \int_{\Sigma_0} \langle d \star j\phi(y) \rangle. \tag{10}$$

Since $\mathrm{Link}(\Sigma_0, y) = 0$, then

$$Q(\Sigma + \partial\Omega_0) = \int_{\Omega} \langle d \star j\phi(y) \rangle = Q(\Sigma). \tag{11}$$

This is why we call these *topological operators*. The unitary operator is defined by taking the exponential of the conserved charge,

$$U(g, \Sigma_{D-1}) = \exp(i\alpha_a Q_a) = \exp\left(i\alpha \int \star j\right), \tag{12}$$

where $g \in G$ is an element of the group G and α_a are the parameters of the transformation. In the case of $U(1)$ we have $g = e^{i\alpha}$. We say then that this is a *topological codimension-1 operator.

We may understand the the action of 0-form symmetries in terms of the unitary operators as well. Recall that the local operators transform under some transformation by

$$U_g(t)\mathcal{O}(x)U_g^{-1}(t) = \mathcal{O}'(x). \tag{13}$$

Where these operators are defined in some spatial slice Σ_{D-1}. The corresponding expression for topological operators is given by

$$\langle U(\Sigma_{D-1})\mathcal{O}(x) \rangle = \langle \mathcal{O}'(x)U(\Sigma'_{D-1}) \rangle. \tag{14}$$

Since operators are defined over a Hilbert space at a constant time, we can not compare operators at different times. Since the manifold Σ_{D-1} need not to be necessarily a spatial slice for general topological operators, the expression only makes sense inside correlation functions, where we can compare operators at different spacetime points without problem. Thus we assume that Σ'_{D-1} is a slice at a different time, say t'. Note that taking the limit $t \to t'$ we recover the usual transformation law. What is happening diagrammatically is shown below.

Let $U(S^{D-1})$ be an operator defined over a $(D-1)$-sphere. As we have seen, these operators act over local operators $\mathcal{O}(x)$ and transforms them into another operator $\mathcal{O}'(x)$ as shown in the figure he operator
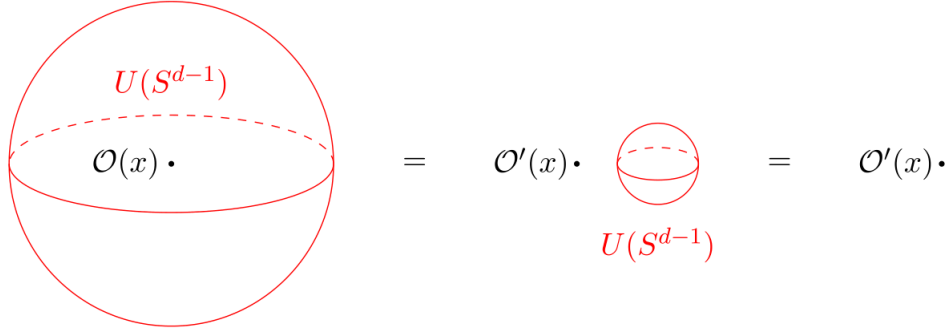

Figure 1: Taken from [1].

$U(S^{D-1})$ links with $\mathcal{O}(x)$ then we deform it until it no longer links. Then we can deform the S^{D-1} to a point and the reminiscent operator is what we call $\mathcal{O}'(x)$. The mathematical expression in this case is

$$U(S^{D-1})\mathcal{O}(x) = \mathcal{O}'(x), \tag{15}$$

where there is no operator in the right-hand side because we deformed it to the identity.

2.1 0-form Groups

We define the 0-form group G^0 to be the group composite of the elements g that parametrizes the operator U_g.

The composition of two topological operators are given by the composition law of the group

$$\langle U_g(\Sigma_{D-1})U_{g'}(\Sigma_{D-1})\rangle = \langle U_{gg'}(\Sigma_{D-1})\rangle. \tag{16}$$

Thus, inserting the composition of topological operators into these correlation functions is equivalent to insert one topological operator with the composition of the elements. This is called the fusion rule,

$$U_g \otimes U_{g'} = U_{gg'}. \tag{17}$$

From now on, we will drop the correlation function notation but it is to be understood that it is there unless it said the contrary. We now give a more formal definition of a 0-form symmetry.

Definition: A 0-form symmetry is a codimension-1 operator that is topological and invertible.

Topological means that

$$U_g(\Sigma_{D-1}) = U_g(\Sigma'_{D-1}), \tag{18}$$

f we can deform Σ_{D-1} continuously into Σ'_{D-1}. The invertibility property follows from the composition rule

$$U_g(\Sigma_{D-1})U_g^{-1}(\Sigma_{D-1}) = 1. \tag{19}$$

As a final result that we will need later, note that for an operator $\mathcal{O}(x)$ with charge q, the Ward identity becomes (without the correlators)

$$\partial_\mu j^\mu(y)\mathcal{O}(y) = q\delta^D(x-y)\mathcal{O}(y), \tag{20}$$

where it was used the fact that $\mathcal{O}(x) \to e^{i\alpha q}\mathcal{O}(x)$. In terms of differential forms,

$$(d \star j)\mathcal{O}(x) = q\delta^D(x)\mathcal{O}(x), \tag{21}$$

where $\delta^D(x)$ is the Poincaré dual of $\delta^D(x-y)$. Thus for an operator $U(S^{D-1})$ acting over a local operator $\mathcal{O}(x)$, we have

$$\begin{aligned}
U(S^{D-1})\mathcal{O}(x) &= \exp\left(i\alpha \int_{S^{D-1}} \star j\right)\mathcal{O}(x) \\
&= \exp\left(i\alpha \int_{\Omega_D} d \star j\right)\mathcal{O}(x) \\
&= \exp\left(i\alpha \int_{\Omega_D} q\delta^D(x)\right)\mathcal{O}(x) \\
&= \exp(i\alpha)\mathcal{O}(x),
\end{aligned} \tag{22}$$

where Ω_D is some D-dimensional manifold whose S^{D-1} is the border.

3 Higher-Form Symmetries

With our formal definition, it is easy to generalize the idea to higher-form symmetries.

Definition: A p-form symmetry is a codimension-$(p+1)$ operator that is topological and invertible.

Let's see why it must be a codimension-$(p+1)$ operator. Consider the variation of the action we've saw before we've performed the gauging of the symmetry,

$$\delta S = \int d^4x \; j_\mu \partial^\mu \Lambda. \tag{23}$$

For a D-dimensional space and in the language of differential forms, this equation becomes

$$\delta S = \int \star j \wedge d\Lambda. \tag{24}$$

Consider that Λ is a p-form, consequently $d\Lambda$ will be a $(p+1)$-form. Then, in order to match the dimension of the spacetime, $\star j$ must be a $(D-p-1)$-form. This implies that

$$Q = \int_{\Sigma_{D-p-1}} \star j, \tag{25}$$

must be integrated over a $(d-p-1)$-dimensional manifold, and since the operator U is defined as the exponential of the charge, it is also defined over a $(d-p-1)$-dimensional manifold, which characterizes it as a codimension-$(p+1)$ operator. The unitary operator is defined as before

$$U_g(\Sigma_{D-p-1}) = \exp\left(i\alpha \int_{\Sigma_{D-p-1}} \star j\right), \tag{26}$$

but now the degree of the form has changed.

We now must look at which objects are charged under those symmetries, that is, which objects the p-form operators act on.

The same rules from before apply here. There is an inverse operator

$$U(\Sigma_{D-p-1})U^{-1}(\Sigma_{D-p-1}) = 1. \tag{27}$$

The group of p-form symmetries G^p consists of the elements that parametrize the operator $U_g(\Sigma_{D-p-1})$. Just as before, the composition of those operators is given by

$$U_g(\Sigma_{D-p-1})U_{g'}(\Sigma_{D-p-1}) = U_{gg'}(\Sigma_{D-p-1}), \tag{28}$$

which is denoted the fusion rule

$$U_g \otimes U_{g'} = U_{gg'}. \tag{29}$$

Moreover, the p-form symmetry group is abelian for $p \geq 1$. This follows because we can deform an operator over another without ever crossing them.

The "usual" symmetries, now understood as 0-form symmetries, act over local operators, whose are supported on 0-dimensional regions. This is something we understand from standard QFT. We say then that the local operators are charged under the 0-form symmetry. We now wish to understand which objects are charged under p-form symmetries. It is straightforward to note that for $p \geq 1$, the p-form symmetry will never act over a local operator because we can always deform the operator in a way that it doesn't intersect with the point. Therefore, the p-form symmetry acts over objects that have a dimension $q \geq p$.

Case $q = p$

For this case, we define an operator $U_g(\Sigma_{D-p-1})$ over a $(D-p-1)$-dimensional manifold that links with some operator $\mathcal{O}(M_p)$ defined over a p-dimensional manifold. Then we deform $U_g(\Sigma_{D-p-1})$ over $\mathcal{O}(M_p)$, leaving behind a local operator $\mathcal{O}(x)$ as before. We suppose that the only local operators that can be inserted along the manifold M_p are multiples of the local identity operator. Thus, we may say that these multiples are complex numbers $\phi(g)$,

$$\phi(g) \in \mathbb{C}^\times, \quad \mathbb{C}^\times \equiv \mathbb{C}\backslash\{\,0\}$$

The action of the operator U_g over $\mathcal{O}$ is represented below The composition of the group induces a composition of those functions $\phi(g)$,

$$\phi(g)\phi(g') = \phi(gg'). \tag{30}$$

Therefore the functions $\phi(g)$ form a 1-dimensional representation of the p-form group G^p. Since G^p is an abelian group, from Schur's Lemma we know that every irreducible representation of this group is 1-dimensional, then the functions $\phi(g)$ form precisely the irreducible representation of the group G^p. Therefore, the operators $\mathcal{O}(M_p)$ transforms under the fundamental representations of the p-form group, which shows that the charged objects under p-form symmetries are extended operators supported on p-dimensional manifolds.
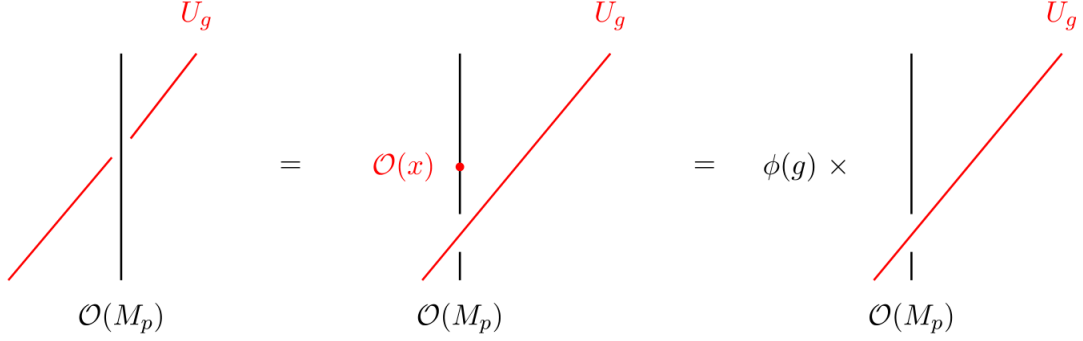
Figure 2: Taken from [1].

We can easily generalize the result we have seen before for the Ward identity in terms of p-forms. For some charged operator $\mathcal{O}(\Sigma_p)$ with charge $q \in \mathbb{Z}$ we have

$$(d \star j)\mathcal{O}(\Sigma_D) = q\delta^{D-p}(\Sigma_p)\mathcal{O}(\Sigma_D) \tag{31}$$

where $\delta^{D-p}(\Sigma_p)$ is the Poincaré dual of the delta function defined on Σ_D. Hence, for extended operators we have

$$U_g(S^{D-1})\mathcal{O}(\Sigma_p) = \exp(i\alpha q)\mathcal{O}(S^{D-1}). \tag{32}$$

To summarize our results, let us study a $U(1)$ gauge theory in D dimensions.

U(1) Gauge Theory in D dimensions

Let A be a 1-form $U(1)$ gauge field with

$$F = dA, \tag{33}$$

where F is a 2-form. In $D =$ this is precisely Maxwell theory. From (33) we have

$$dF = d \star (\star F) = 0. \tag{34}$$

Therefore the Noether current then is the $(D-2)$-form, $\star F$. The conserved charge is

$$Q^m = \int_{\Sigma_2} \star(\star F) = \int_{\Sigma_2} F, \tag{35}$$

and the unitary operator is

$$U_g^m(\Sigma_2) = \exp\left(i\alpha \int_{\Sigma_2} F\right). \tag{36}$$

The index m appears because we call this the *magnetic symmetry*. For $p \geq 1$-forms we integrate the charge over a $D - p - 1$ manifold. Since this is 2-dimensional manifold, $D - p - 1 = 2 \Rightarrow p = D - 3$. Thus, the magnetic symmetry is related to a $(D-3)$-form symmetry.

The other conservation law follows from the "Maxwell" equations,

$$d \star F = 0, \tag{37}$$

which implies that in this case the conserved current is the 2-form, F. The conserved charge is

$$Q^e = \int_{D-2} \star F, \tag{38}$$

And the unitary operator

$$U_g^e(\Sigma_{D-2}) = \exp\left(i\alpha \int_{D-2} \star F\right). \tag{39}$$

This is the *electric symmetry*. In this case, $D - p - 1 = D - 2 \Rightarrow p = 1$. Then, the electric symmetry is a 1-form symmetry, independent of the dimension.

Therefore, this $U(1)$ gauge theory has 2 higher-form symmetries, an *electric 1-form symmetry* and a *magnetic $(D - 3)$-form symmetry*. We represent then the symmetry group as

$$U(1)_e \times U(1)_m. \tag{40}$$

The charged objects under the electric 1-form symmetry are the *Wilson lines* $W_q(L)$,

$$W_q(L) = \exp\left(iq \int A_L\right), \tag{41}$$

where q is the charge of the Wilson line. Using (31) we have

$$(d \star F)W_q(L) = q\delta^{D-1}(L)W_q(L). \tag{42}$$

This shows that the electric 1-form symmetry measures the charge of the Wilson line.

The charged objects under the magnetic $(D - 3)$-form symmetry are the *'t Hooft lines* $T(\Sigma_{D-3})$. Using the same idea, we can show that the magnetic $(D - 3)$-form symmetry measures the magnetic charge of the 't Hooft line.

4 Discrete Gauge Symmetries

Discrete gauge symmetries are related to symmetry transformations that belongs to discrete groups, rather continuous. It can be better understood in terms of higher-form symmetries. Before we understand how this works, we need to introduce two concepts, the Pontryagin dual and the Screening of operators.

4.1 Pontryagin Dual

For some p-form symmetry group G^p we may define the homomorphism

$$\phi : G^{(p)} \to U(1). \tag{43}$$

These maps form a group,

$$\phi(g)\phi'(g) = (\phi\phi')(g) \in U(1). \tag{44}$$

We denote this group by

$$\widehat{G}^{(p)} = \{\text{all homomorphisms } G^{(p)} \to U(1)\}. \tag{45}$$

Let us ilustrate this idea with two examples.

Example: $G^p = U(1)$

Define the homomorphism as,

$$\phi(g) = g^q = e^{i\alpha q}, \ \alpha \in [0, 2\pi), \ q \in \mathbb{Z}. \tag{46}$$

Let

$$\phi_1(g) = g, \tag{47}$$
$$\phi_2(g) = g^2 \tag{48}$$
$$\phi_3(g) = g^3. \tag{49}$$

The composition gives,

$$\phi_1(g)\phi_2(g) = \phi_3(g). \tag{50}$$

This is the composition rule of the $\mathbb{Z}$ group. Therefore the Pontryagin dual for $U(1)$ is

$$\widehat{G}^{(p)} = \mathbb{Z}. \tag{51}$$

Example: $G^p = \mathbb{Z}_N$

In this case, $g = e^{\frac{2\pi i \alpha}{N}}$. Defining the homomorphism as

$$\phi = g^q = e^{\frac{2\pi i \alpha q}{N}}. \tag{52}$$

Take $N = 3$ and let

$$\phi_1(g) = g \tag{53}$$
$$\phi_2(g) = g^2. \tag{54}$$

The composition gives

$$\phi_1(g)\phi_2(g) = \phi_1(g). \tag{55}$$

Which is the composition rule of $\mathbb{Z}_N$. Therefore

$$\widehat{G}^{(p)} = \mathbb{Z}_N. \tag{56}$$

That is, the Pontryagin dual of $\mathbb{Z}_N$ is the group itself.

Double Pontryagin Duality

Since

$$\phi : G^{(p)} \to U(1), \tag{57}$$

we can take

$$\varphi : \widehat{G}^{(p)} \to U(1). \tag{58}$$

Note that $\phi(g) \in U(1)$ and $\varphi(\phi(g)) \in (1)$, then

$$\widehat{\widehat{G}^{(p)}} = G^{(p)}. \tag{59}$$

4.2 Screening

Definition 1 *A $p \geq 1$-dimensional operator $\mathcal{O}_p$ can be screened into another operator $\mathcal{O}'_p$ if there exists a $(p-1)$-dimensional operator $\mathcal{O}_{p-1}$ that can be inserted between O_p and O'_p.*
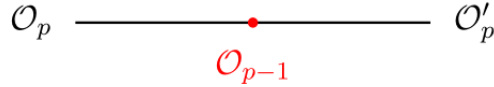

Figure 3: Taken from [1].

If $\mathcal{O}_p$ can be screened to the identity we say that it was completely screened.

Screening implies in the same charges. Note that if two operators $\mathcal{O}_p$ and $\mathcal{O}'_p$ are screened, the can be topologically deformed into one another. This implies that the charges measured by the symmetry operator must be the same.

References

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